Orientation

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There are two notions of orientation on manifolds that one needs to be able to go between.

1 Homology

We recall here that an orientation on a closed manifold at a point $x \in M^n$ is a generator for the "local homology"

$$H_n(M^n|x) = H_n(M^n, M^n - \{x\}) \cong \mathbb{Z}.$$

An orientation is then a consistent assignment

$$\mu : M \to \cup_{x \in M} H_n(M|x)$$

of generators, where the consistency condition is that around each x there is a chart U and a generator

$$\mu_U \in H_n(M|U)$$

such that for every $y \in U$ we have that under the natural map

$$H_n(M|U) \to H_n(M|y)$$

μ_U is sent to μ_y.

2 Tangent Spaces

We define the equivilence relation on bases of vector spaces by saying they are equivilent if the change of basis matrix has positive determinant. There are only two equivilence classes for any (finite dimensional) vector space. An orientation of a vector space is a choice of one of these classes.

If we are considering tangent spaces and we fix a chart then there is the one represented by the standard basis one is represented by the standard basis $\partial/\partial x_i$ and the other is given by the negative standard basis.

An orientation of a vector bundle $B \to M$ is a choice of orientation μ_p for ever B_p satisfying the following compatability condition: every local trivilisation $U \subseteq M$

$$\varphi : \pi^{-1}(U) \to U \times V$$

preserves orientations fiberwise when $\mathbb{R}^n$ is given the standard orientation. A manifold is oriented if its tangent space is oriented.

Lemma. *A smooth n-manifold is orientable iff it has a no-where-zero n-form.*

3 Comparing

3.1 De Rham

If we consider De Rham cohomology and use universal coefficients theorem for homology / cohomology with field valued coefficients then we get that

$$\mathbb{R} \cong H_n(M|x;\mathbb{R}) \cong H^n_{dR}(M)$$

thus choosing a generator / basis for the LHS is exactly choosing a non-zero element of the De Rham cohomology, which is a no-where-zero n-form.

3.2 Exponential

It is a fundamental result that every smooth manifold has a Riemannian metric. Given a Riemannian metric one can define an exponential map that is a local diffeomorphism

$$U \subseteq TM \to V \subseteq M$$

Given $v \in T_pM$ then by existence uniqueness of ODE's there is a unique (local) geodesic $\gamma_\nu : [0,1] \to M$ starting at p, i.e. $\gamma(0) = p$. The exponential at p is then

$$\nu \mapsto \gamma_\nu(1)$$

Clearly locally the differential of this map is the identity and hence a local isomorphism on the tangent spaces and hence a local diffeomorphism.

Remark: We know that in general

$$\text{dim}TM = 2\text{dim}M$$

and so it is clear that we must be taking a subset of the tangent bundle in a strict sense.

3.3 Orientation on a simplex

This is explained in Milnor-Stasheff. An element of $H_n(M|x)$ is just a (class of) singular n-chain (linear combo of simplicies) whose boundary lies in $M - \{x\}$, in particular there is a representative in every class that is given by a single simplex $\sigma : \Delta^n \to M$ why?. If we choose this simplex to be a differentiable embedding why can we do that? differentiable, embedding. This would make more sense if we took the interior of the cell? At least then its a manifold. then we can pushforward the cannonical orientation on Δ^n given to it by being a subspace of $\mathbb{R}^n$. Thus after picking a generator we can push it forward and get an orientation on the tangent space. one then has to check that this is an orientation. https://math.stackexchange.com/questions/43779/equivalent-definitions-of-orientation

4 Extending to the Boundary

4.1 Homology

Now if M has a boundary then the local homology at any point on the boundary is just 0. Thus we define an orientation of a manifold with boundary to just be an orientation on $M - \partial M$. The orientation of the boundary becomes meaningless when we consider it *as a boundary*. However the orientation on the manifold *induces* an orientation on the boundary *as a manifold* (not a submanifold). Precisely if we let $M^n = \partial W^{n+1}$ we have a commuting diagram from the LES

$$\begin{array}{ccccccc} H_{n+1}(M) & \longrightarrow & H_{n+1}(W) & \overset{\sim}{\hookrightarrow} & H_{n+1}(W,M) & \overset{\sim}{\longrightarrow} & H_n(M) \\ \| & & \updownarrow \sim & & & & \updownarrow \sim \\ 0 & & H^0(W) & & \overset{\sim}{\longleftrightarrow} & & H^0(M) \end{array}$$

where the verticle arrows are isomorphisms by Poincare duality and $H_{n+1}(M) = 0$ by the dimension of M. This zero implies the injectivity of the next arrow horrozontally. The bottom horrozontal arrow is an iso because....? Which implies that the top arrows are iso by the commutativity of the diagram. This shows that a fundamental class, orchoosing a generator of $H_{n+1}(W)$, through these cannonical isos gives a generator of $H_n(M)$, thus an orientation of W induces an orientation on the boundary.

4.2 Tangent Space

If $x \in M$ is a point on the boundary then we define the tangent space at x by an infinitesimal thickening of the boundary. The result is that

$$TW_x \cong T\partial W_x \oplus \mathbb{R}$$

where the real component is identified with "an outward pointing normal".